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Perturbation of Nonlinear Potential Problems*

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Invariant imbedding can be viewed as a nonclassical perturbation technique which converts nonlinear boundary-value problems to quasilinear initial-value problems. The application of these ideas to the nonlinear partial differential equations of the type arising in potential theory is investigated.

1. INTRODUCTION

Many phenomena arising in potential theory can be described by partial differential equations of the form

$$u_{xx} + u_{yy} + g(u) = 0, \tag{1}$$

where (1) holds in some given region and is subject to prescribed conditions on the boundary of the region. The case where g is linear in u can be solved numerically by a number of methods [1]. Recently much interest has been centered on the conversion of the linear version of (1), together with its boundary conditions, to an equivalent initial value problem [2]. One of the basic techniques employed has been invariant imbedding which is based upon varying the region of interest, a nonclassical perturbation technique [3].

This paper will investigate the application of imbedding to nonlinear partial differential equations. We will begin by showing how nonlinear

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ordinary differential equations subject to two point boundary conditions can be converted to initial value problems. Then we will try to extend the formalism to nonlinear partial differential equations. At all times we will be endeavoring to obtain a feasible computational method. We will deal only with simple examples in a purely formal manner. Rigorous derivations of the results will require many functional analytic concepts which will be left to subsequent papers.

2. Ordinary Differential Equations [3]

We will consider the nonlinear ordinary differential equation,

$$u_{xx} + g(u) = 0, \tag{2}$$

subject to the two point boundary conditions

$$u(0) = 0 \quad \text{and} \quad u_x(a) = c. \tag{3}$$

Let us assume that (1) and (2) have a unique solution for all $a \in [0, b]$ for some b. We will regard u as a function of a and c, i.e.,

$$u = u(x, a, c), \tag{4}$$

for $-\infty < c < \infty, 0 \leqslant a \leqslant b$. Suppose we fix x and perturb c. Differentiating (2) we find

$$u_c = (\partial/\partial c)\, u(x, y, c) \tag{5}$$

satisfies

$$(u_c)_{xx} + g_u u_c = 0. \tag{6}$$

Differentiating the boundary conditions of (3) yields

$$u_c(0, a, c) = 0, \qquad u_c(a, a, c) = 1. \tag{7}$$

We now repeat the procedure, perturbing a instead of c. The perturbation of (2) gives us the result

$$(u_a)_{xx} + g_u u_a = 0, \tag{8}$$

and the first boundary condition yields

$$u_a(0, a, c) = 0. \tag{9}$$

Differentiating the second boundary condition we have

$$(\partial/\partial a)\, u_x(a, a, c) = (\partial/\partial a)\,(c) = 0, \tag{10}$$

and

$$(\partial/\partial a)\, u_x(a, a, c) = u_{ax}(a, a, c) + u_{xx}(a, a, c). \tag{11}$$

Hence, by (2) evaluated at $x = a$, (10), and (11),

$$[u_a(a, a, c)]_x = g(u(a, a, c)). \tag{12}$$

We see that u_a and u_c satisfy similar linear boundary value problems and in fact they are related by the equation,

$$u_a(x, a, c) = g(u(a, a, c))\, u_c(x, a, c), \tag{13}$$

which is the basic result we will use.

Let us define $r(a, c)$ by

$$r(a, c) = u(a, a, c). \tag{14}$$

Then differentiating, we have

$$r_a = u_x(a, a, c) + u_a(a, a, c). \tag{15}$$

The first term of the right is given by the boundary conditions, (3), while the second term is given by (13) evaluated for $x = a$. Thus we find r satisfies

$$r_a = c + g(r)\, r_c\,, \tag{16}$$

and since the first boundary condition of (3) holds for all $a \in [0, b]$, (16) is subject to the initial condition

$$r(0, c) = 0. \tag{17}$$

A few observations can be made immediately. It is clear that r satisfies an initial value problem and since (13) can be rewritten as

$$u_a(x, a, c) = g(r)\, u_c(x, a, c), \tag{18}$$

u satisfies an intial value problem in a and c for a fixed x. The case where g is linear in u is particularly nice since

$$r(a, c) = R(a)\, c, \tag{19}$$

and R satisfies a Riccati equation. Since (16) is quasilinear, many methods are available for its numerical solution. Of particular interest are methods which linearize g since then the separation of variables technique in (19) can be carried out [4].

3. Partial Differential Equations

Let us now try to carry out the same type of perturbation on the nonlinear partial differential equation,

$$u_{xx} + u_{yy} + g(u) = 0, \tag{20}$$

on the rectangle, $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant 1$, subject to the boundary conditions

$$u(0, y) = u(x, 0) = u(x, 1) = 0, \tag{21}$$

and

$$u_x(a, y) = f(y). \tag{22}$$

Again we assume that (20)–(22) have a unique solution for all $a \in [0, b]$. We start by regarding u as not only a function of x and y but also of the scalar parameter a and the function f, i.e.,

$$u = u(x, y, a, f). \tag{23}$$

Differentiating (20) with respect to a yields

$$(u_a)_{xx} + (u_a)_{yy} + g_u u_a = 0, \tag{24}$$

and from (21),

$$u_a(0, y, a, f) = u_a(x, 0, a, f) = u_a(x, 1, a, f) = 0. \tag{25}$$

Since (22) can be written as

$$u_x(a, y, a, f) = f(y), \tag{26}$$

differentiation with respect to a gives the boundary condition

$$[u_a(a, y, a, f)]_x = - u_{xx}(a, y, a, f), \tag{27}$$

or by (20)

$$[u_a(a, y, a, f)]_x = u_{yy}(a, y, a, f) + g(u(a, y, a, f)). \tag{28}$$

We must now see what the solution changes are when the function f is perturbed. As in [5], we will use a functional derivative. Suppose for some suitable class of functions the limit,

$$D(u, f, w) = \lim_{\Delta \to 0} \frac{u(x, y, a, f) - u(x, y, a, f + \Delta w)}{\Delta}, \tag{29}$$

exists. It is a direct exercise to verify the D is given by

$$D(u, f, w) = \int_0^1 u_f(x, y, s)\, w(s)\, ds, \tag{30}$$

where u_f satisfies

$$(u_f)_{xx} + (u_f)_{yy} + g_u u_f = 0, \tag{31}$$

subject to

$$u_f(0, y, s) = u_f(x, 0, s) = u_f(x, 1, s) = 0, \tag{32}$$

and

$$[u_f(a, y, s)]_x = \delta(y - s), \tag{33}$$

where δ is the Dirac delta function.

The function u_f is the functional derivative of u with respect to f. We see from (31) that u_f depends on u and thus also depends on a and f. We write

$$u_f = u_f(x, y, a, s, f). \tag{34}$$

Comparing the linear system (24), (25), and (26) with the linear system (31), (32), and (33), we immediately recognize the following relationship

$$u_a(x, y, a, f) = \int_0^1 [u_{yy}(a, s, a, f) + g(u(a, s, a, f))]\, u_f(x, y, a, s, f)\, ds. \tag{35}$$

This equation is the multidimensional analog of (13). We define $r(a, y, f)$ as

$$r(a, y, f) = u(a, y, a, f), \tag{36}$$

so that (35) becomes

$$u_a(x, y, a, f) = \int_0^1 [r_{yy}(a, s, f) + g(r(a, s, f))]\, u_f(x, y, a, s, f)\, ds, \tag{37}$$

a linear initial value problem in a ($\geqslant x$) subject to the initial condition

$$u(x, y, x, f) = r(x, y, f), \tag{38}$$

and subject to (21). We now have to find an equation for r and we are done. Differentiating (36) we have

$$r_a(a, u, f) = u_x(a, y, a, f) + u_a(a, y, a, f), \tag{39}$$

and by (22) and (37)

$$r_a(a, y, f) = f(y) + \int_0^1 [r_{yy}(a, s, f) + g(r(a, s, f))]\, r_f(a, y, s)\, ds, \tag{40}$$

where r_f is defined as

$$r_f(a, y, s) = u_f(a, y, a, s, f), \tag{41}$$

and is determined by (31)–(33) with $x = a$. Finally the initial conditions for (40) are given by (21)

$$r(0, y, f) = r(a, 0, f) = r(a, 1, f) = 0. \tag{42}$$

As with ordinary differential equations, we have a quasilinear initial value problem. The main difficulty is that we are now dealing with functions of two variables and a function. The case where g is linear is particularly elegant since then we can show that the solution of (31)–(33) is sufficient to give a solution of the problem. The details of the linear case can be found in [2]. Once again methods based on the linearization of g should prove effective. Finally we note that many nonstandard numerical methods should follow from (40) via the use of various quadrature formulas. This topic will be investigated in the future.

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